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A New Biorthogonal Wavelet with Very Simple Coefficients

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Abstract Constructing wavelets via the lifting scheme with indeterminate coefficient has been proposed recently. The constructed lifting schemes generally have four steps. Differently we use the five-step lifting scheme to construct wavelet. Finally, we obtain a wavelet with very simple coefficients especially in its lifting scheme. Its order of vanishing moments is the same with that of D9/7 wavelet, but the coefficients of the corresponding filters are simple fractions. In its lifting scheme, the denominators of all coefficients are the power of 2 and the numerators are 1. Every step of the lifting algorithm only needs a shift operation with 0 to 3 bits. This wavelet approaches or exxceeds D9/7 wavelet in some mathematical properties, which will be verified in the numerical experiments of image compressions.

Keywords lifting scheme, vanishing moments, biorthogonal wavelets

一个新的带非常简单系数的双正交小波

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摘　要　近来，通过带待定系数提升格式构造小波已经被提出，构造出的提升格式一般有 4 步提升。不同地是，本文使用 5 步提升格式来构造小波。最终获得一个带非常简单系数（特别是在其提升格式中）的小波。它的消失矩阶数与 D9/7 小波一样大，但相应滤波器的系数都是简单分数。在它的提升格式中，所有系数的分母是 2 的幂次，并且分子为 1。提升格式的每一步只需要一个 0 到 3 位的位移。此小波在某些数学性质上接近或超过了 D9/7 小波，这将被之后的图像压缩的数值实验所验证。

关键词　提升格式　消失矩　双正交小波

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1 Introduction

In recent years, as a new wavelet construction method[1,2] and a fast implementation of Mallat algorithm[3,4], the lifting scheme has held many researchers' interest. It has been highly accepted as the recommended wavelet transform algorithm in JPEG2000 standard.

In the Daubechies's classic method to construct biorthogonal wavelets[5], the order of vanishing moments of synthesis and analysis wavelets must be specified at first. In this way there are usually few unordinary (real number) solutions. Recently a new method which constructs wavelet via the lifting scheme with indeterminate coefficients has been proposed[6,7]. In this method, the element metrics with indeterminate coefficients are assumed firstly. Secondly, the values of the indeterminate coefficients are calculated under some constrained conditions. In this way, the factorizations of the polynomial are avoided, but solving a linear system is necessary which is more easier and more flexible. In

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[6] the proposed wavelets are the existing wavelets (the CDF set). Liang *et al* have [7] explored this method and successfully constructed several wavelets.

In the paper, we increase the number of steps of the lifting scheme to explore more wavelets which have some advantages over D9/7 wavelet.

2 The Lifting Scheme

In this paper our notations are $z = e^{-i\omega}$, and $h(z) = \sum_k h_k z^k$, $h_e(z) = \sum_k h_{2k} z^k$, $h_o(z) = \sum_k h_{2k+1} z^k$. And $h(z)$ is the Fourier transform of the synthesis low-pass filter, $\tilde{h}(z)$ filter is the Fourier transform of the analysis low-pass filter; $g(z)$ is the Fourier transform of the synthesis high-pass filter, and $\tilde{g}(z)$ is the Fourier transform of the analysis high-pass filter. p and $\tilde{p}$ denote the order of vanishing moments of analysis and synthesis wavelets respectively.

Note $\begin{bmatrix} \tilde{h}_e(z) & \tilde{g}_e(z) \\ \tilde{h}_o(z) & \tilde{g}_o(z) \end{bmatrix}$ as the multiphase matrix in the analysis process. The Mallat algorithm can be represented as:

$$\begin{bmatrix} lp_e(z) \\ hp_e \end{bmatrix} = \tilde{P}(z^{-1})^{\mathrm{T}} \begin{bmatrix} x_e(z) \\ zx_o(z) \end{bmatrix}$$

The lifting scheme can be written as the product of the element matrices [3], i. e. Lifting step (S-type):

$$\tilde{P}^{\mathrm{new}}(z) = \tilde{P}(z) \begin{bmatrix} 1 & 0 \\ -s(z^{-1}) & 1 \end{bmatrix}$$

Dual lifting step (W-type):

$$\tilde{P}^{\mathrm{new}}(z) = \tilde{P}(z) \begin{bmatrix} 1 & -t(z^{-1}) \\ 0 & 1 \end{bmatrix}$$

3 Constructing Biorthogonal Wavelets via the Lifting Scheme

The element matrices of D5/3 and D9/7 wavelet[3,4] have so-called 'three-item-adder' form, i. e.

$$\begin{bmatrix} 1 & 0 \\ s(z) & 1 \end{bmatrix}, \text{ where } s(z) = \alpha(1+z),$$

$$\begin{bmatrix} 1 & t(z) \\ 0 & 1 \end{bmatrix}, \text{ where } t(z) = \alpha(1+z^{-1}).$$

The three-item-adder form has the small computational complexity, and holds "in place" property naturally.

The general constructing process is listed as follows:

(1) In the synthesis process, write the polyphase matrix as the product of element matrices:

$$P(z) = \begin{bmatrix} h_e(z) & h_o(z) \\ g_e(z) & g_o(z) \end{bmatrix} = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \cdots \begin{bmatrix} 1 & \alpha_1(1+z^{-1}) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \alpha_0(1+z) & 1 \end{bmatrix}$$

(2) Multiply all the element matrices and obtain the expressions of $h(z)$ and $g(z)$. Under conditions of the vanishing moments or other conditions, a set of equations can be constructed.

For instance, $\frac{\mathrm{d}}{\mathrm{d}z^i} h^i(z) \mid_{z=-1} = 0$, $0 \leqslant i < p$, $\frac{\mathrm{d}}{\mathrm{d}z^i} g^i(z) \mid_{z=1} = 0, 0 \leqslant i < \tilde{p}$; $h(1) = 2$; $g(-1) = (-1)\tilde{h}(1) = -1$ and so on.

(3) Increase p and $\tilde{p}$ step by step, and try to solve the equations until p and $\tilde{p}$ can not be increased any more.

(4) In the analysis process, let $\tilde{\alpha}_i = -\alpha_i$, we finally have

$$\tilde{P}(z) = \begin{bmatrix} 1 & 0 \\ \alpha_0(1+z) & 1 \end{bmatrix} \begin{bmatrix} 1 & -\alpha_1(1+z^{-1}) \\ 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1/K_1 & 0 \\ 0 & 1/K_2 \end{bmatrix}$$

In [7], the general 9/7 taps wavelets via the lifting scheme is S-W-S-W type. There is only one solution with irrational number under the conditions:

$$\frac{\mathrm{d}}{\mathrm{d}z^i} h^i(z) \mid_{z=-1} = 0, i = 0,1,2,3$$

$$\frac{\mathrm{d}}{\mathrm{d}z^i} g^i(z) \mid_{z=1} = 0, i = 0,1,2,3, \text{i. e. D9/7}$$

wavelet. The other 9/7 taps wavelets constructed in [7] only have (4,2) vanishing moments.

How to construct wavelets with (4,4) vanishing moments via lifting scheme? In [7], authors changed three-item-adder to more-item-adder in the element matrix. Instead, we adhere to five-step lifting scheme with three-item-adder in every element matrix. It is W-S-W-S-W type as follows:

$$P(z) = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix} \begin{bmatrix} 1 & \alpha_4(1+z^{-1}) \\ 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & \alpha_1(1+z^{-1}) \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \alpha_0(1+z) & 1 \end{bmatrix}$$

Restricting by conditions:

$$\frac{\mathrm{d}}{\mathrm{d}z^i} h^i(z) \mid_{z=-1} = 0, \ \frac{\mathrm{d}}{\mathrm{d}z^i} g^i(z) \mid_{z=1} = 0, \ i = 0,1,$$

2,3, $h(1)=2$, $g(-1)=-1$, we acquired two unordinary solutions:

$$\begin{cases}\alpha_0=1/4,\alpha_1=-1,\alpha_2=-1/8,\alpha_3=\alpha_4=1/2\\ K_1=K_2=1\end{cases}$$

$$\begin{cases}\alpha_0=-5/4,\alpha_1=-1,\alpha_2=1/16,\alpha_3=\alpha_4=2\\ K_1=-2,K_2=-1/2\end{cases}$$

The first solution is:

$$h(z)=\frac{1}{128}z^{-5}+\frac{1}{32}z^{-4}-\frac{3}{128}z^{-3}+\frac{32}{64}z^{-1}+\frac{15}{16}+\cdots$$

$$g(z)=\frac{1}{64}z^{-3}+\frac{1}{16}z^{-2}-\frac{1}{8}z^{-1}-\frac{5}{16}+\frac{23}{32}z-\cdots$$

The second solution is:

$$h(z)=\frac{25}{128}z^{-5}-\frac{5}{32}z^{-4}-\frac{75}{128}z^{-3}+\frac{3}{4}z^{-2}+\frac{57}{64}z^{-1}-\frac{3}{16}+\cdots$$

$$g(z)=-\frac{5}{64}z^{-3}+\frac{1}{16}z^{-2}+\frac{1}{4}z^{-1}-\frac{5}{16}+\frac{5}{32}z-\cdots$$

As we know, even the filters satisfied the perfect reconstruction condition, φ and $\tilde{\varphi}$ which are defined as follows:

$$\hat{\varphi}(\omega)=\prod_{p=1}^{+\infty}\frac{h(2^{-p}\omega)}{\sqrt{2}},\hat{\tilde{\varphi}}(\omega)=\prod_{p=1}^{+\infty}\frac{\tilde{h}(2^{-p}\omega)}{\sqrt{2}}$$

would be likely to diverge in $L^2(R)$[5]. The second solution just diverge in $L^2(R)$, we do not discuss on it any more. Next, we name the wavelet according to the first solution as FLS9/11 (Five lifting scheme) wavelet for its support length. The figure 1 plots the scaling and wavelet funcitons.

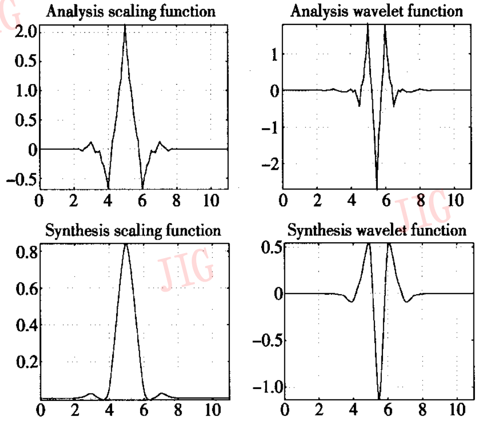

Fig. 1　Analysis scaling and wavelet (upper), synthesis scaling and wavelet(lower)

4　The Lifting Scheme of FLS9/11 Wavelet

The analysis multiphase matrix of the first solution is presented as follows:

$$\tilde{P}(z)=\begin{bmatrix}1 & -(1+z^{-1})/4\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 1+z & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & (1+z^{-1})/8\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -(1+z)/2 & 1\end{bmatrix}\begin{bmatrix}1 & -(1+z^{-1})/2\\ 0 & 1\end{bmatrix}$$

Because $\tilde{h}(1)=1, h(1)=1$, we add the normalized multiplier factor $\begin{bmatrix}\sqrt{2} & 0\\ 0 & \sqrt{2}/2\end{bmatrix}$ at their end.

FLS9/11 wavelet can generate a reversible integer transform[4]. According to the 2-D normalization method in [8] we proposed the 2-D algorithm only with shift and adder as follows:

(1) Horizontal transform:

$$s_l=x_{2l},d_l=x_{2l+1}$$

$$d_l=d_l-(s_l+s_{l+1})>>2,s_l=s_l+d_l+d_{l-1}$$

$$d_l=d_l+(s_l+s_{l+1})>>3,s_l=s_l-(d_l+d_{l-1})>>1$$

$$d_l=d_l-(s_l+s_{l+1})>>1$$

(2) Vertical transform: Repeat the process of (1) to columns, and left shift every sample in LL, HL, LL, HH with one bit.

Compared with the 1-D analysis lifting scheme of D9/7 wavelet [3]:

$$s_l=x_{2l},d_l=x_{2l+1}$$

$$d_l=d_l+\alpha(s_l+s_{l+1}),s_l=s_l+\beta(d_l+d_{l-1})$$

$$d_l=d_l+\gamma(s_l+s_{l+1}),s_l=s_l+\delta(d_l+d_{l-1})$$

$$s_l=\zeta s_l,d_l=d_l/\zeta$$

Where:

$$\alpha=-1.586\,134\,342,\beta=-0.052\,980\,118\,54,$$

$$\gamma=0.882\,911\,076\,2,\delta=0.443\,506\,852\,2,$$

$$\zeta=1.149\,604\,398.$$

Obviously, FLS9/11 wavelet has a much lower computational complexity than D9/7 wavelet. The number of adders of FLS9/11 wavelet is the same as that of D9/7 wavelet. However, there are only 6 shifts with 0 to 3 bits in FLS9/11 wavelet (considering the scaling process in 2-D case), but 6 float-point multiplications in D9/7 wavelet. If we use fixed-point multiplications with finite precise in the lifting scheme of D9/7 wavelet (for example, the above factors are left shifted with 13 bits before the multiplication, the results are right shifted with 13 bits after the multiplication), the complexity of D9/7 wavelet will be reduced, but still much higher than that of FLS9/11 wavelet and the compression performance will decline rapidly[9].

In [10], the authors proposed a multiplierless implementation of D9/7 wavelet. However, they only implemented the Mallat algorithm without the lifting scheme. Their methods are based on the multiplication which is performed using shifts and additions after approximating each coefficient as a sum or difference of powers of two (SPT). Every their proposed methods need total 32 numbers of SPT, i. e. there must be at least 32 shifts.

The VLSI architecture implementation of D9/7 wavelet has also been studied, for example in [11]. But their focus is on memory requirement, critical path etc. instead of computational complexity.

There is another advantage of FLS9/11 wavelet. It can also be used in the reversible integer transform, but D9/7 wavelet cannot.

5 Some Mathematical Properties

We recur to some important conceptions in the wavelet theory[5,12] to analyze the mathematical properties of FLS9/11 wavelet. An important characteristic of a scaling function is its smoothness in the sense of degree of differentiability. In this paper, we calculate the smoothness of functions with Holder regularity and Sobelev regularity. The Riesz Bounds measures the redundancy of a Riesz base. In [12] Unser defined a projection angle and an asymptotic approximation constant to characterize the biorthogonal wavelets system. We compared FLS9/11 wavelet with D9/7 wavelet in table 1. All the methods we mentioned have been described in [12]. Compared with D9/7 wavelet, FLS9/11 wavelet has a looser set of Riesz bounds. This indicates that the orthogonality of FLS9/11 wavelet is less than that of D9/7 wavelet, which is consistent with the fact that the projection angle between the synthesis and analysis subspaces of FLS9/11 wavelet is greater than that of D9/7 wavelet. On the other hand, the regularity of the synthesis scaling function of FLS9/11 wavelet is higher, but that of analysis scaling function is smaller. At last FLS9/11 wavelet has a smaller asymptotic approximation constant than D9/7 wavelet. For above reasons, FLS9/11 wavelet would have a very close performance to D9/7 wavelet in image compression.

Tab. 1 Some mathematical properties

	D9/7 analysis	D9/7 synthesis	FLS 9/11 analysis	FLS 9/11 synthesis
Holder regularity	1.07	1.70	0.881 5	2.020 1
Sobolev regularity	1.41	2.12	1.241 4	2.412 7
Asymptotic approximation constant	-	0.006 96	-	0.006 15
Riesz down bounds	0.926	0.943	1	0.523 8
Riesz up bounds	1.065	1.084	1.931 8	1
Projection cosines	0.983 87	-	0.967 73	-

6 The Coding Gain

The transform coding gain is a measure of the coding performance of transform-based coding schemes. The coding gain is defined as the ratio of the reconstruction error variance of the transform coding scheme and the reconstruction error variance of a pulse code modulation (PCM) scheme. In [13], authors derive their expression for the coding gain as follows:

$$G_{SBC}(\rho) = \frac{1}{\prod_{k=1}^{M} (A_k B_k)^{\alpha_k}}$$

Where

$$A_k = \sum_i \sum_j h_k(i) h_k(j) \rho^{|j-i|}$$

$B_k = \sum_j g_k(i)^2$, M is the number of subbands (now it is equal to 2), $h_k(i)$ ($k = 1, 2$) are the coefficients of the k-th analysis filter, $g_k(i)$ ($k = 1, 2$) are the coefficients of the k-th synthesis filter, and ρ is the correlation of the source modeled as a one-dimensional Markov source.

The figure 2 plots the coding gains of D9/7 wavelet and FLS9/11 wavelet. The coding gain of FLS9/11 wavelet is close to that of D9/7 wavelet and may even

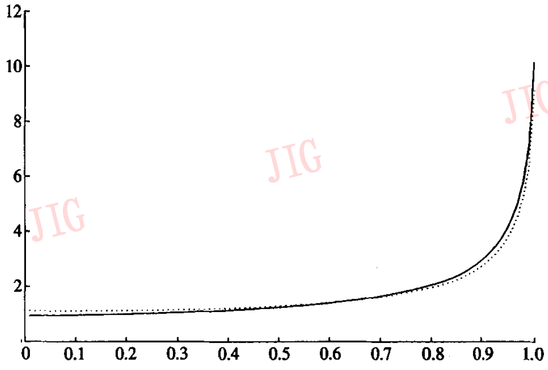

Fig. 2 Coding gain with $0 \leq \rho \leq 1$, dash line shows D9/7 wavelet, real line shows FLS9/11 wavelet

exceed that of D9/7 wavelet when $\rho > 0.6407$.

7　Numerical Experiments

We do some compression experiments on the images which are all of 512 × 512 size and 8 bits depth. We employ EZW coding method in our experiments. SWE13/7 wavelet is the wavelet constructed by Sweldens in [1] which has (4,4) vanishing moments. In table 2, "Float" means floating-point transforms and "Integer" means transforms mapping integer to integer for D9/7 wavelet, integer transforms for FLS9/11 wavelet and SWE13/7 wavelet. The maximal analysis level is 6. The results are SNR values.

In table 2, the compression performance of FLS9/11 wavelet approaches that of D9/7 wavelet even exceeds in 0.1250 bpp, which is consistent with our mathematic analysis. The performance of FLS9/11 wavelet exceeds that of SWE13/7 wavelet in most cases.

Tab. 2　Comparison of compression performance in PSNR. The upper is for Lena, the lower is for Goldhill

		0.1250	0.2500	0.5000	1.0000
Lena	D9/7 Float	30.4060	33.3438	36.4238	39.5816
	D9/7 Integer	30.3360	33.1946	36.0999	38.8608
	FLS 9/11 Float	30.6175	33.5889	36.4560	39.4235
	FLS 9/11 Integer	30.6159	33.5035	36.3618	39.0667
	SWE13/7 Float	30.5660	33.4876	36.4685	39.6853
	SWE13/7 Integer	30.5586	33.4724	36.4379	39.5403
Goldhill	D9/7 Float	28.0417	30.2839	32.5935	35.5928
	D9/7 Integer	28.0114	30.2198	32.3295	35.1305
	FLS 9/11 Float	28.0674	29.7225	31.9775	35.1312
	FLS 9/11 Integer	28.0870	29.7102	31.8870	35.0601
	SWE13/7 Float	27.9683	29.6147	31.8976	35.1124
	SWE13/7 Integer	27.9671	29.6072	31.8909	35.0683

8　Conclusion

The performance of FLS9/11 wavelet exceeds that of SWE13/7 wavelet in most cases. Moreover, for the very low computational complexity, FLS9/11 wavelet is a efficient wavelet for the image transform coding, and is specially competent for the low efficient computation environment(such as the chips only supporting adder and shift) and VLSI architecture.

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